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On the chromatic number of $(P_5, dart)$-free graphs. (English. English summary)
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Let $\omega(G)$ and $\chi(G)$ be the clique number and chromatic number, respectively, of the graph G. Call a family $\mathcal{G}$ of graphs "hereditary" if, for any $G\in\mathcal{G}$, all induced subgraphs of G are also in $\mathcal{G}$. For a collection of graphs $\{H_1,\dots,H_k\}$ and a graph G, say that G is $(H_1,\dots,H_k)$-free if G does not contain any induced subgraph isomorphic to H_i for any $1\le i\le k$. Note that the collection of all $(H_1,\dots,H_k)$-free graphs is a hereditary family for any $H_1,\dots,H_k$.

A "χ-binding function" for a hereditary family $\mathcal{G}$ is a function f such that $\chi(G)\le f(\omega(G))$ for all $G\in\mathcal{G}$. If such a function exists, we also say that the family is "χ-bounded". It is known that if H contains a cycle, then the class of H-free graphs is not χ-bounded [see P. Erdős and A. Hajnal, Acta Math. Acad. Sci. Hungar. **17** (1966), 61–99; MR0193025]. An important conjecture independently due to A. Gyárfás [Zastos. Mat. **19** (1987), no. 3-4, 413–441 (1988); MR0951359] and D. P. Sumner [in *The theory and applications of graphs (Kalamazoo, Mich., 1980)*, 557–576, Wiley, New York, 1981; MR0634555] says that if H does not contain a cycle, then the class of H-free graphs is χ-bounded. This conjecture has been verified only for certain cases, such as when H is a path P_k on k vertices. A related conjecture due to B. A. Reed [J. Graph Theory **27** (1998), no. 4, 177–212; MR1610746] says that for every graph G, we have $\chi(G)\le\lceil(\omega(G)+\Delta(G)+1)/2\rceil$, where $\Delta(G)$ is the maximum degree of G. Reed's conjecture is also largely open, having been verified only for certain classes of graphs, such as the class of $(P_5, dart)$-free graphs [I. Schiermeyer, Discrete Math. **339** (2016), no. 7, 1940–1943; MR3488929], which is the focus of the current work.

In this paper, the authors prove that the class of $(P_5, dart)$-free graphs is χ-bounded, with $\chi(G)\le\frac{3}{4}\omega(G)^2$, and the class of $(P_5,C_5,dart)$-free graphs is χ-bounded with $\chi(G)\le\binom{\omega(G)+1}{2}$. The former improves the χ-binding function $f(\omega)=\omega^2$ found by Schiermeyer, and the latter gives a tighter bound as compared to the one by M. Chudnovsky and V. Sivaraman [J. Graph Theory **90** (2019), no. 1, 54–60; MR3879962] for the class of (P_5,C_5)-free graphs, namely $f(\omega)=2^{\omega-1}$. Here, a *dart* is a graph on five vertices obtained by adding a pendant vertex attached to any vertex of degree 3 in K_4^- (the graph obtained from K_4 by deleting an edge).

The proof is by induction on the clique number ω, and assumes that a minimal counterexample exists in order to derive a contradiction. It is shown that such a minimal counterexample G must have an induced P_4 such that every vertex is either in this path or in its neighborhood. The vertex set of G is then partitioned in such a way that χ-binding functions or clique numbers can be computed for the induced subgraphs on these parts, using the assumption that G is *dart*-free. This data is then put together to derive a contradiction.

Of particular use is the following observation: if G is a connected P_5-free graph that is not complete, and if C is a minimal cut set for G, then for any $c\in C$ there is at most one connected component T of $G-C$ such that the induced subgraph on $T\cup\{c\}$ is not complete bipartite. For the sake of simplicity, floors and ceilings are not mentioned in any of the statements or proofs, but can be inferred from context. A typo: in the abstract of the paper, probably the work of Schiermeyer was meant to be cited, instead of [L. Esperet et al., Discrete Math. **313** (2013), no. 6, 743–754; MR3010736]. However, the latter work is also cited within the paper for the following result: The current best known bound for the class of P_5-free graphs is given in the work of Esperet et al., namely $\chi(G)\le 5\cdot 3^{\omega(G)-3}$.

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